**GOSFORD HIGH SCHOOL**
**EXTENSION TWO MATHEMATICS**
**ASSESSMENT TASK 3 JUNE 2008**
**TIME ALLOWED: ONE AND A HALF HOURS.**

**Question 1.**

**a)** Evaluate $\int_0^{\frac{\pi}{8}} \cos^2 2x\,dx$ **2.**

**b)** By the use of partial fractions find $\int \frac{x+1}{x^3+x^2-6x}\,dx$ **3.**

**c)** Find $\int \sin^4 x.\cos^5 x\,dx$ **3.**

**d)** By using the substitution $x = a\tan\theta, (0 < a)$ find $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$ **3.**

**e)** Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x}\sqrt{1-x}}$ **3.**

**f)** Find $\int \frac{dx}{\sqrt{28-12x-x^2}}$ **3.**

**g)** **i)** Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ **3.**

**ii)** Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$ **1.**

**h)** If $I_n = \int x^n e^{-2x}\,dx$ prove $I_n = \frac{-x^n e^{-2x}}{2} + \frac{n}{2} I_{n-1}$ and hence find $\int x^2 e^{-2x}\,dx$ **4.**

**i)** i) Use the substitution $u = \pi - x$ to show that

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx .$$ **2.**

ii) Hence deduce that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$ **3.**

**Question 2.**

**a)** The area between the curve $y = 8x - x^2$, the $x$ axis and the line $x = 4$ is rotated about the line $x = 4$. Find the volume generated by using:

i) slicing **3.**

ii) cylindrical shells. **3.**

**b)** Find the volume of the solid generated when the area between the curves $y = x^2$ and $y = 4x$ is rotated about the line $x = 5$. **4.**

**c)** A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Find the volume of the solid formed if every section perpendicular to the major axis is an isosceles triangle with altitude 6. **4.**

**Question 3.**

Let $P(cp, \frac{c}{p})$ be a point on the rectangular hyperbola $xy = c^2$.

**i)** Show that the equation of the tangent at $P$ is given by $x + p^2 y = 2cp$. **2.**

**ii)** Show that the area between the asymptotes and the tangent at $P$ is a constant. **2.**

**iii)** If $Q\left(cq, \frac{c}{q}\right)$ is another point on the hyperbola and the tangents at the points $P$ and $Q$ meet at the point R $(x_0, y_0)$ prove that

$$p + q = \frac{2c}{y_0} \quad \text{and} \quad pq = \frac{x_0}{y_0}$$ **4.**

**iv)** If the length of the chord $PQ$ is $d$ units, show that

$$d^2 = c^2(p-q)^2\left\{1 + \frac{1}{p^2q^2}\right\}$$ **2.**

**v)** If $d$ remains fixed, deduce that the locus of R has the equation
$4c^2(x^2 + y^2)(c^2 - xy) = x^2y^2d^2$. **3.**

ASSESSMENT 2. Ext 2. JUNE 2008.

Q1) a) $\int_0^{\pi/8} \cos^2 2x \, dx$

$= \frac{1}{2}\int_0^{\pi/8} \cos 4x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right]_0^{\pi/8}$

$= \frac{1}{2}\left[\left(\frac{1}{4} + \frac{\pi}{8}\right) - 0\right]$

$= \frac{1}{8} + \frac{\pi}{16}$

b) $\frac{x+1}{x^3+x^2-6x} = \frac{x+1}{x(x+3)(x-2)}$

let

$\frac{x+1}{x(x+3)(x-2)} = \frac{a}{b} + \frac{b}{x+3} + \frac{c}{x-2}$

$\therefore x+1 = a(x+3)(x-2) + bx(x-2) + cx(x+3)$

let $x=0$: $1 = -6a$

$a = -\frac{1}{6}$

let $x = 2$: $3 = 10c$

$c = \frac{3}{10}$

let $x=-3$: $-2 = 15b$

$b = \frac{-2}{15}$

$\therefore \int \frac{x+1}{x^3+x^2-6x} \, dx = \int \frac{-1}{6x} \frac{-2}{15(x+3)} + \frac{3}{10(x-2)} \, dx$

$= -\frac{1}{6}\ln x - \frac{2}{15}\ln(x+3) + \frac{3}{10}\ln(x-2) + c$

c) $\int \sin^4 x \cdot \cos^5 x \, dx$

$= \int \sin^4 x \, (\cos^2 x)^2 \cdot \cos x \, dx$

$= \int \sin^4 x \, (1-\sin^2 x)^2 \cdot \cos x \, dx$

$= \int \sin^4 x \, (1 - 2\sin^2 x + \sin^4 x) \cdot \cos x \, dx$

$= \int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cdot \cos x \, dx$

let $u = \sin x$

$du = \cos x$

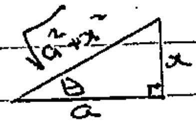
$du = \cos x \, dx$

$= \int u^4 - 2u^6 + u^8 \, du$

$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$

$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$

d) $\int \frac{dx}{(a^2+x^2)^{3/2}}$




let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta \, d\theta$

$= \int \frac{a\sec^2\theta \, d\theta}{(a^2 + a^2\tan^2\theta)^{3/2}}$

$= \int \frac{a\sec^2\theta \, d\theta}{(a^2(1+\tan^2\theta))^{3/2}}$

$= \int \frac{a\sec^2\theta \, d\theta}{(a^2\sec^2\theta)^{3/2}}$

$= \int \frac{a\sec^2\theta}{a^3\sec^3\theta} \, d\theta$

$= \frac{1}{a^2}\int \frac{1}{\sec\theta} \, d\theta$

$= \frac{1}{a^2}\int \cos\theta \, d\theta$

$= \frac{1}{a^2}\sin\theta + C$

$= \frac{1}{a} \cdot \frac{x}{\sqrt{a^2+x^2}} + C$

e) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{x}.\sqrt{1-x}}$

Let $x = u^2$ $\quad x = 1/2,\ u = \frac{1}{\sqrt{2}}$

$\frac{dx}{du} = 2u$ $\quad x = 0,\ u = 0$

$dx = 2u\,du$

$\displaystyle\int_0^{1/\sqrt{2}} \frac{2u\,du}{\sqrt{u^2}.\sqrt{1-u^2}}$

$\displaystyle= \int_0^{1/\sqrt{2}} \frac{2u\,du}{u\sqrt{1-u^2}}$

$\displaystyle= 2\int_0^{1/\sqrt{2}} \frac{du}{\sqrt{1-u^2}}$

$= 2\left[\sin^{-1} u\right]_0^{1/\sqrt{2}}$

$= 2\left(\sin^{-1} \frac{1}{\sqrt{2}} - \sin^{-1} 0\right)$

$= 2 \times \frac{\pi}{4}$

$= \frac{\pi}{2}.$

f) $\displaystyle\int \frac{dx}{\sqrt{28-12x-x^2}}$

$\displaystyle= \int \frac{dx}{\sqrt{28-(x^2+12x)}}$

$\displaystyle= \int \frac{dx}{\sqrt{28-(x^2+12x+36-36)}}$

$\displaystyle= \int \frac{dx}{28-((x+6)^2-36)}$

$\displaystyle= \int \frac{dx}{\sqrt{64-(x+6)^2}}$

$= \sin^{-1}\left(\frac{x+6}{8}\right) + c.$

g) (i) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

$t = \tan\frac{x}{2}$ $\quad : \quad x = \pi/2,\ t = 1$

$x = 0,\ t = 0.$

$\displaystyle= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$\displaystyle= \int \frac{2}{1+t^2+2t}$

$\displaystyle= \int_0^1 \frac{2}{(t+1)^2}\,dt.$

$= \left[\frac{-2}{t+1}\right]_0^1$

$= -1 - (-2)$

$= 1.$

ii) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\sin x}\,dx$

$\displaystyle= \int_0^{\pi/2} \frac{\sin x+1-1}{1+\sin x}\,dx$

$\displaystyle= \int_0^{\pi/2} \frac{1+\sin x}{1+\sin x} - \frac{1}{1+\sin x}\,dx$

$\displaystyle= \int_0^{\pi/2} 1\,dx - \int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$

$= \left[x\right]_0^{\pi/2} - 1$

$= \frac{\pi}{2} - 1.$

b) $I_n = \int x^n . e^{-2x}\,dx$

$= \int x^n \left(\frac{d}{dx}\left(-\tfrac{1}{2}e^{-2x}\right)\right) dx$

$= \frac{-x^n e^{-2x}}{2} - \int -\tfrac{1}{2}e^{-2x} . nx^{n-1}\,dx$

$= \frac{-x^n e^{-2x}}{2} + \frac{n}{2}\int x^{n-1} e^{-2x}\,dx$

$= \frac{-x^n e^{-2x}}{2} + \frac{n}{2} I_{n-1}.$

$\therefore\ I_2 = \int x^2 e^{-2x}\,dx$

$I_2 = \frac{-x^2 e^{-2x}}{2} + I_1$

$I_1 = \frac{-x e^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$

$= \frac{-x e^{-2x}}{2} - \frac{1}{4}e^{-2x}$

$\therefore\ I_2 = \frac{-x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{1}{4}e^{-2x}$

$= -\tfrac{1}{4}e^{-2x}\left(2x^2 + 2x + 1\right)$

(i) (i) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

$u = \pi - x \qquad x = \pi : u = 0$

$\frac{du}{dx} = -1 \qquad x = 0 : u = \pi$

$du = -dx$

$\int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)} - du.$

$= \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\,du.$

$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx.$

(ii) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

$= \int_0^{\pi} \frac{\pi \sin x - x\sin x}{1+\cos^2 x}\,dx$

$= \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$\therefore\ 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx$

let $u = \cos x \qquad x = 0.\ u = 1$

$\frac{du}{dx} = -\sin x \qquad x = \pi\ \ u = -1$

$du = -\sin x\,dx$

$= \pi\int_1^{-1} \frac{-du}{1+u^2}$

$= \pi\int_{-1}^{1} \frac{du}{1+u^2}$

$= \pi\left[\tan^{-1} u\right]_{-1}^{1}$

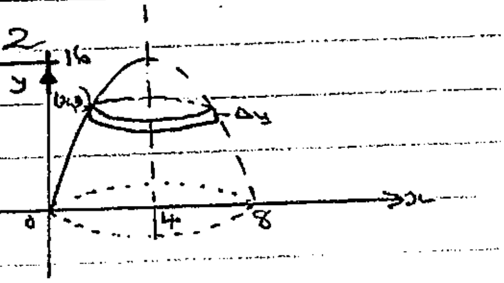
$= \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$\frac{\pi^2}{2}$

$\therefore\ \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

Question 2

a)




i) Volume of slice $= \pi(4-x)^2\,\Delta y$

$V \doteq \sum_{y=0}^{16} \pi\,(4-x)^2\,\Delta y.$

$V = \pi\int_0^{16} (4-x)^2\,dy$

$$V = \pi\int_0^{16} 16 - 8x + x^2 \, dy$$

$$= \pi\int_0^{16} 16 - (8x - x^2) \, dy$$

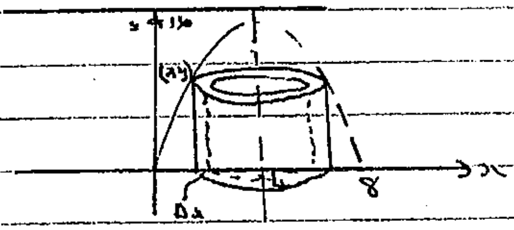
$$= \pi\int_0^{16} 16 - y \, dy$$

$$= \pi\left[16y - \frac{y^2}{2}\right]_0^{16}$$

$$= \pi\left[(256 - 128) - 0\right]$$

$$= 128\pi \text{ cubic units}$$

(ii)



Volume of shell $= 2\pi r h$

$$= 2\pi(4 - x)y\,\Delta x$$

$$V \doteqdot \sum_{x=0}^{4} 2\pi(4 - x)y\,\Delta x$$

$$V = \int_0^4 2\pi(4 - x)y \, dx$$

$$= 2\pi\int_0^4 (4 - x)(8x - x^2) \, dx$$

$$= 2\pi\int_0^4 32x - 4x^2 - 8x^2 + x^3 \, dx$$

$$= 2\pi\int_0^4 32x - 12x^2 + x^3 \, dx$$

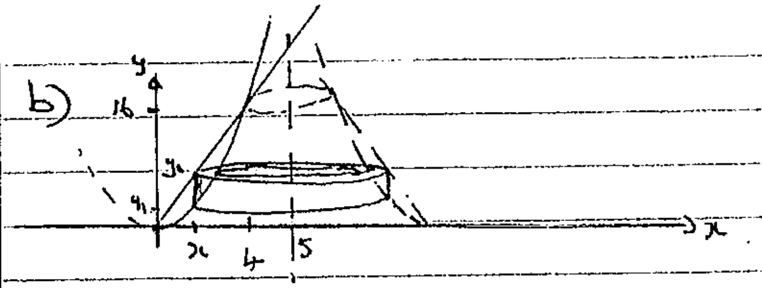
$$= 2\pi\left[16x^2 - 4x^3 + \frac{x^4}{4}\right]_0^4$$

$$= 2\pi\left[(256 - 256 + 64) - 0\right]$$

$$= 2\pi(64)$$

$$= 128\pi \text{ cubic units.}$$

b)



Volume of shell $= 2\pi r h$

$$= 2\pi(5 - x)(y_2 - y_1)\Delta x$$

$$= 2\pi(5 - x)(4x - x^2)\Delta x$$

$$\text{Volume} \doteqdot \sum_{x=0}^{4} 2\pi(5 - x)(4x - x^2)\Delta x$$

$$V = 2\pi\int_0^4 (5 - x)(4x - x^2) \, dx$$

$$= 2\pi\int_0^4 20x - 9x^2 + x^3 \, dx$$

$$= 2\pi\left[10x^2 - 3x^3 + \frac{x^4}{4}\right]_0^4$$

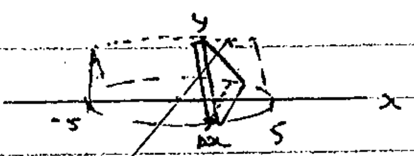
$$= 2\pi\left[(160 - 192 + 64) - 0\right]$$

$$= 2\pi(32)$$

$$= 64\pi \text{ cubic units.}$$

N.B. if used slicing

$$V = \pi\int_0^{16} 10\sqrt{y} - \frac{7y}{2} + \frac{y^2}{16} \, dy.$$



Volume of slice $= \frac{1}{2}(2y) \times 6 . \Delta x$

$$= 6y\,\Delta x$$

$$V \doteqdot \sum_{x=-5}^{5} 6y\,\Delta x$$

$$V = \int_{-5}^{5} 6y\,\Delta x$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$y = 4\sqrt{1 - \frac{x^2}{25}}$$

$$y = \frac{4}{5}\sqrt{25 - x^2}$$

$$= \int_{-5}^{5} 6y \, dx.$$

$$= \frac{24}{5}\int_{-5}^{5} \sqrt{25 - x^2} \, dx. \rightarrow \text{(semi circle)}$$

$$= \frac{24}{5} \times \frac{1}{2} \times \pi \times 5^2 = 60\pi \text{ cubic units.}$$

Question 3

i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dx}{dy} = \frac{-c^2}{x^2}$

at $x = cp$ : $\frac{dy}{dx} = -\frac{1}{p^2}$

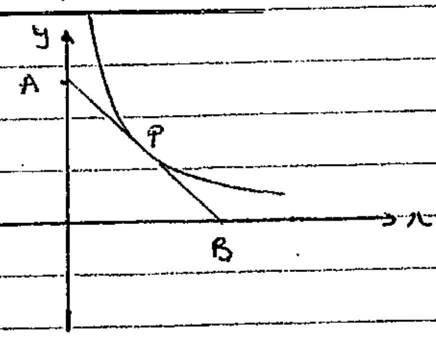
$\therefore$ eqn tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$

ii)




$x + p^2y = 2cp$

'x' intercept $(y=0)$ : $x = 2cp$

'y' intercept $(x=0)$ : $y = \frac{2c}{p}$

Area $= \frac{1}{2}bh$

$= \frac{1}{2} \times 2cp \times \frac{2c}{p}$

$= 2c^2$

which is a constant.

iii) $x + p^2y = 2cp$ ---(1)

$x + q^2y = 2cq$ ---(2)

(1) − (2) $(p^2 - q^2)y = 2c(p-q)$

$y = \frac{2c(p-q)}{(p+q)(p-q)}$

ie $y_0 = \frac{2c}{p+q}$ ---(A)

Sub into (1)

$x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$x = 2cp - \frac{2cp^2}{p+q}$

$= \frac{2cp(p+q) - 2cp^2}{p+q}$

$x_0 = \frac{2cpq}{p+q}$ ---(B)

from (A) $(p+q)y_0 = 2c$

$p+q = \frac{2c}{}$

Now $\frac{x_0}{y_0} = \frac{\frac{2cpq}{p+q}}{\frac{2c}{p+q}}$

$= \frac{2cpq}{p+q} \times \frac{p+q}{2c}$

$= pq$

iv) $d^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$= c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$= c^2(p-q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$

$= c^2(p-q)^2 + \frac{c^2(q-p)^2}{p^2q^2}$

$d^2 = c^2(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)$

v) $d^2 = c^2(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)$

$= c^2\left((p+q)^2 - 4pq\right)\left(1 + \frac{1}{(pq)^2}\right)$

from above $pq = \frac{x}{y}$ , $p+q = \frac{2c}{y}$

$d^2 = c^2\left(\left(\frac{2c}{y}\right)^2 - \frac{4x}{y}\right)\left(1 + \frac{1}{\frac{x^2}{y^2}}\right)$

$$d^2 = c^2\left(\frac{4c^2-4xy}{y^2}\right)\left(1+\frac{y^2}{x^2}\right)$$

$$d^2 = 4c^2\frac{(c^2-xy)}{y^2}\times\left(\frac{x^2+y^2}{x^2}\right)$$

$$x^2y^2d^2 = 4c^2(c^2-xy)(x^2+y^2)$$